Equilibrium theory of coherent vortex and zonal jet formation in a system of nonlinear Rossby waves

Peter B. Weichman
BAE Systems, Advanced Information Technologies, 6 New England Executive Place, Burlington, Massachusetts 01803, USA
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The problem of coherent vortex and zonal jet formation in a system of oceanic or planetary nonlinear Rossby waves is considered from the point of view of the late time steady state achieved by free decay of a given initial state. Statistical equilibrium equations respecting all conservation laws are constructed for a broad class of models, generalizing those derived previously for two-dimensional inviscid Euler flow. Jetlike solutions are ubiquitous, with large coherent vortices existing only when there is a background flow whose velocity locally cancels the beta effect.

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The formation of jets (localized, elongated energetic flows) in rotating two-dimensional (2D) flow, such as planetary atmospheres and oceans, is a ubiquitous phenomenon. Although jetlike structures are often dictated by external forcing, or by boundary constraints, zonal (east-west) jets may also form via unforced evolution of essentially isotropic initial conditions in large systems where boundaries are unimportant [1]—the remarkable band structure of Jupiter is a famous example. Although the planetary rotation axis clearly distinguishes the zonal and meridional directions, its effect on the dynamics is rather subtle, and there is no simple argument why this should lead to highly elongated structures. In particular, conserved quantities like energy E and enstrophy Ω_2, as well as nonlinear advection effects, remain isotropic.

Recently, a new, highly anistropic adiabatic invariant B, special to systems of interacting Rossby waves [2], has been argued to provide the required dynamical mechanism [3]. In particular, for the ever larger scale flows generated by the well-known inverse cascade of energy, B increases strongly with scale if the energy spectrum remains isotropic. Conservation of B concentrates the spectrum in the meridional direction, leading to zonally concentrated flows. Well defined Rossby waves exist only at midlatitudes, so this offers only a partial explanation since jet formation is also seen in near-equatorial flows.

In this work a very general, complementary approach is considered. Rather than trying to follow the very complicated turbulent dynamics from some given initial condition, one seeks to understand only the range of possible long-time steady states produced by free decay of such flows. In particular, *equilibrium* steady states are considered, which are completely specified by the values of all the conserved quantities. In standard three-dimensional (3D) thermodynamic systems, often only energy and particle number are conserved. In the 2D flows of interest, there are an infinite number, providing a far greater range of interesting equilibria with nontrivial spatial structure [4–11].

In the following, an exact nonlinear PDE that determines the equilibrium flows is derived, and solved in various limits. It will be shown, quite generally, that the latitude dependence of the Coriolis force (the "beta effect") destabilizes large coherent vortex structures: the only stable equilibrium flows are those that are zonally translation invariant, i.e., jetlike. Only in a background flow that cancels or locally overturns the beta effect are coherent vortices stable. Interestingly, conservation of B, which requires a beta effect, also fails in this limit, and therefore does not hinder the formation of such structures. The equilibrium arguments are independent of all dynamical considerations, and hence, do not depend on the existence of B, but the fact that they are so consistent suggests a relation. This will be a subject of future investigation.

For generality, consider a class of equations (examples to follow) which may be written in the form

$$\partial_t Q + \mathbf{v} \cdot \nabla Q = 0. \quad (1)$$

The incompressible velocity field $\mathbf{v}=\nabla \times \psi \equiv (\partial_y \psi, -\partial_x \psi)$. The stream function ψ is assumed related to the convectively conserved "charge field" Q through an energy functional $\mathcal{H}[Q]$:

$$\psi(\mathbf{r}) = \delta\mathcal{H}/\delta Q(\mathbf{r}). \quad (2)$$

An example is the Charney-Hasagawa-Mima (CHM) equation in which $Q=\omega+k_R^2\psi+f$, where $\omega=\nabla\times\mathbf{v}\equiv\partial_x v_y-\partial_y v_x =-\nabla^2\psi$ is the vorticity. The Coriolis function $f =2\Omega_E \sin[\theta(y)]$, where $\theta(y)$ is the latitude, and $\Omega_E =2\pi/(24\text{ h})$ is the rotation frequency of the Earth, and coordinates are chosen so that the x-axis points eastward and the y-axis northward. In this case

$$\mathcal{H} = \frac{1}{2}\int d^2r \int d^2r' [Q(\mathbf{r})-f(\mathbf{r})]g(\mathbf{r},\mathbf{r}')[Q(\mathbf{r}')-f(\mathbf{r}')], \quad (3)$$

in which g is the Green function of the operator $-\nabla^2+k_R^2$ satisfying free slip and/or periodic boundary conditions. In the beta-plane approximation, $f=f_0+\beta_f y$, where $f_0=2\Omega_E \sin(\theta_0)$, $\beta_f=2(\Omega_E/R_E)\cos(\theta_0)$, where R_E is Earth's radius, and θ_0 is a reference latitude. The Rossby radius of deformation is $R_0=1/k_R=c/f$, where $c\sim 3$ m/s is the speed of internal gravity waves. In principle $k_R=k_R(y)$, but usually one sets $k_R=1/R_0(0)$. Only β_f then produces anisotropy. The free space Green function is the modified Bessel function,

$g(\mathbf{r},\mathbf{r}')=K_0(k_R|\mathbf{r}-\mathbf{r}'|)/2\pi$, and linearizing (1) produces the usual Rossby wave dispersion relation $\omega=-\beta_f k_x/(k_R^2+k^2)$. The CHM equation (reducing to the Euler equation for $k_R=0$) is an approximation to the shallow water equations in which propagating gravity waves are neglected. The surface height is then proportional to ψ, and adiabatically follows the flow. This equation also describes drift wave plasmas, in the 2D plane perpendicular to an applied magnetic field, where ψ is the electrostatic potential and ω the charge density.

For later purposes, it is useful to define the Legendre transform of $\mathcal{H}$,

$$\mathcal{L}[\psi]=\mathcal{H}[Q]-\int d^2r\psi(\mathbf{r})Q(\mathbf{r}), \tag{4}$$

in which (2) is used to substitute ψ for Q. The relation may be inverted via the implied identity

$$Q(\mathbf{r})=-\delta\mathcal{L}/\delta\psi(\mathbf{r}). \tag{5}$$

From the example (3) one obtains the simple result

$$\mathcal{L}=-\int d^2r\left[\frac{1}{2}|\nabla\psi|^2+\frac{1}{2}k_R^2\psi^2+f\psi\right], \tag{6}$$

the first two terms of which may be recognized as the kinetic and potential energies.

Since $\mathbf{v}\cdot\nabla\psi\equiv 0$, it follows that $\mathcal{H}$ is conserved [12]. With periodic or free slip boundary conditions, it follows also from (1) that $\Omega_h=\int d^2rh[Q(\mathbf{r})]$ is conserved for any one-dimensional (1D) function $h(\sigma)$, conveniently summarized by

$$g(\sigma)=\int d^2r\delta[\sigma-Q(\mathbf{r})], \tag{7}$$

conserved for any σ. One recovers $\Omega_h=\int d\sigma h(\sigma)g(\sigma)$. Certain "momentum functionals"

$$P=\int d^2r\lambda(\mathbf{r})Q(\mathbf{r}) \tag{8}$$

are also conserved if $\mathcal{H}$ has appropriate translation symmetries [13]. For rotational symmetry, the conserved (vertical component of) angular momentum corresponds to $\lambda=\frac{1}{2}r^2$. For translation symmetry along direction $\hat{\mathbf{l}}$, the conserved linear momentum corresponds to $\lambda=\hat{\mathbf{l}}\times\mathbf{r}$.

For the CHM equation in the open beta-plane there is an additional *adiabatically* conserved quantity, which to quadratic order in ψ takes the Fourier space form [2]

$$B=\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\hat{b}(\mathbf{k})\hat{\mathcal{Q}}(-\mathbf{k})\hat{\mathcal{Q}}(\mathbf{k}), \tag{9}$$

in which $\hat{\mathcal{Q}}(\mathbf{k})=(k^2+k_R^2)\hat{\psi}(\mathbf{k})$ is the Fourier transform of $Q-f=(-\nabla^2+k_R^2)\psi$. The key properties are [3] (1) $\hat{b}(\mathbf{k})=\mathcal{O}[(k_R/k)^6]$ decays rapidly for $k/k_R\gg 1$, and (2) for $k/k_R\ll 1$, $\hat{b}(\mathbf{k})=\mathcal{O}(1)$ for $\mathbf{k}$-space directions $\pi/3<\hat{\theta}<2\pi/3$ (but vanishing for $|\hat{\theta}|\to\pi/2$) while $\hat{b}(\mathbf{k})=\mathcal{O}(k_R/k)$ otherwise. Property (1) implies that B, even more so than $\mathcal{H}$, is a large scale quantity, insensitive to small scale variations in Q. Property (2) then implies that the inverse cascade must focus the small k part of the spectrum close to $|\hat{\theta}|=\pi/2$, leading (through the curl relation with $\mathbf{v}$) to large scale zonal flows [3].

When $k_R\to 0$ or $\beta_f\to 0$, the adiabatic conditions fail, and B is no longer conserved [2], so this form should be used only at midlatitudes. Since B is not exactly conserved, it will not be included explicitly in the equilibrium calculation, but its consistency with various proposed equilibrium states will be confirmed at the end.

All statistical equilibrium information is contained in the free energy

$$\mathcal{F}=-\frac{1}{\beta}\ln\left\{\int DQ\Delta_g[Q]e^{-\beta(\mathcal{H}[Q]-\alpha_0 P[Q])}\right\}, \tag{10}$$

in which $\beta=1/T$ is the inverse temperature, α_0 is a Lagrange multiplier for P, $\int DQ$ is a functional integral over all possible configurations of Q, and $\Delta_g[Q]=\Pi_\sigma\delta\{g(\sigma)-g_\sigma[Q]\}$ represents the infinite product of delta functions required to impose the chosen values $g(\sigma)$ of all the conserved integrals $g_\sigma[Q]$ represented by the right hand side of (7). If $Q(\mathbf{r})\to Q_i$ is discretized on a grid with microscopic spacing a, then Liouville's theorem specifies the measure $\int DQ=\lim_{a\to 0}\Pi_i\int_{-\infty}^{\infty}dQ_i$ [5].

It transpires that $\mathcal{F}$ may be computed exactly under the very general assumption that $\mathcal{H}$ and P are insensitive to very small scale fluctuations in Q [4,5]. Due to fine-scale mixing, the equilibrium Q will fluctuate wildly from grid point to grid point, but its average $Q_0(\mathbf{r})$ over a small area $l^2\gg a^2$ (with $l\to 0$ also at the end), will vary smoothly. If $n_0(\mathbf{r},\sigma)$ is the equilibrium probability density for finding a parcel of fluid with charge σ in the area l^2 about $\mathbf{r}$. Then $Q_0(\mathbf{r})=\int\sigma d\sigma n_0(\mathbf{r},\sigma)$. It is assumed that $\mathcal{H}$ is smooth on scale l, and hence, $\mathcal{H}[Q]=\mathcal{H}[Q_0]$. In (3) this is provided by the smoothness of g: although it diverges at the origin, the logarithmic singularity is sufficiently weak that this assumption remains valid [5]. By integrating out the small-scale fluctuations, which may be treated as independent from grid point to grid point, one obtains an entropy contribution,

$$S=\frac{1}{a^2}\int d^2rd\sigma n_0(\mathbf{r},\sigma)\ln[n_0(\mathbf{r},\sigma)], \tag{11}$$

in terms of which the free energy is $\mathcal{F}[n_0]=\mathcal{H}[Q_0]-TS[n_0]$. One now observes that if TS is to remain finite as $a\to 0$ (so that nontrivial equilibria are obtained in which entropy and energy compete), one must adopt the scaling $T=\bar{T}a^2$, $\beta=\bar{\beta}/a^2$, with fixed $\bar{\beta}=1/\bar{T}$.

To self consistently determine $n_0(\mathbf{r},\sigma)$, Lagrange multipliers are introduced via the Gibbs free energy

$$\mathcal{G}=\mathcal{F}-\int d^2r\mu(\sigma)n_0(\mathbf{r},\sigma), \tag{12}$$

in which $\mu(\sigma)$ is used to tune $g(\sigma)$. One may now freely extremalize $\mathcal{G}$ over all n_0, constrained only by the normalization $\int d\sigma n_0(\mathbf{r},\sigma)=1$, to obtain

$$n_0(\mathbf{r},\sigma) = e^{-\bar{\beta}W(\Psi_0-\alpha)}e^{\bar{\beta}[\mu(\sigma)-\sigma(\Psi_0-\alpha)]}, \tag{13}$$

where $\Psi_0(\mathbf{r}) = \delta\mathcal{H}/\delta Q_0(\mathbf{r})$ is the equilibrium stream function, $\alpha(\mathbf{r}) = \alpha_0\lambda(\mathbf{r})$, and

$$W(\tau) = \frac{1}{\bar{\beta}}\ln\left\{\int d\sigma e^{\bar{\beta}[\mu(\sigma)-\sigma\tau]}\right\}. \tag{14}$$

Substituting (13) back into (12) and using (4), finally determines $\mathcal{G}$ as a functional of Ψ_0:

$$\mathcal{G}[\Psi_0] = \mathcal{L}[\Psi_0] - \int d^2r W(\Psi_0 - \alpha). \tag{15}$$

The latter is finally determined by the (minimum for $\bar{T}>0$, maximum for $\bar{T}<0$) condition

$$-\delta\mathcal{L}/\delta\Psi_0(\mathbf{r}) \equiv Q_0(\mathbf{r}) = F[\Psi_0(\mathbf{r}) - \alpha(\mathbf{r})], \tag{16}$$

where $F(\tau) = -\partial_\tau W(\tau)$. Since $W(\tau)$ is convex, $F(\tau)$ is monotonic. Using (5) and (6), the left hand side is $Q_0 = (-\nabla^2 + k_R^2)\Psi_0 + f$, and (16) represents a nonlinear PDE for Ψ_0. The conserved quantities are set by the derivatives $P = -\partial\mathcal{G}/\partial\alpha_0 = \int d^2rF(\Psi_0-\alpha)$, $g(\sigma) = -\delta\mathcal{G}/\delta\mu(\sigma) = \int d^2rn_0(\mathbf{r},\sigma)$.

Substituting (16) into (1), one finds $Q(\mathbf{r},t) = Q_0(\mathbf{r}+\hat{\mathbf{l}}\alpha_0 t)$ or $Q_0(r,\theta+\alpha_0 t)$, depending on the choice of λ. Fixed momentum solutions are not in general static, but require a background flow with velocity α_0 along the symmetry direction.

Equations (15) and (16) are the fundamental results of this paper, providing a complete description of the fluid equilibria for a rather general class of problems. They are equivalent, in the appropriate limits, to previously derived forms for the inviscid Euler equation [4–6], and the quasigeostrophic equations [7,10,11]. Ultimately one is interested in coherent vortex formation, where Q is large in some compact region, and is much smaller outside of it. One sees from (3) that these are high energy configurations, and hence, correspond to $\bar{T}<0$ [5].

For tractable applications "finite level systems" are useful: let $g(\sigma) = \sum_{k=1}^{S} A_k\delta(\sigma-\sigma_k)$, in which A_k is the total area occupied by charge σ_k. One correspondingly requires only a finite number of Lagrange multipliers μ_k: $e^{\bar{\beta}\mu(\sigma)} = \sum_{k=1}^{S} e^{\bar{\beta}\mu_k}\delta(\sigma-\sigma_k)$, and $W(\tau)$ becomes (the log of) a discrete sum. One obtains $n_0(\mathbf{r},\sigma) = \sum_{k=1}^{S}\rho_k(\mathbf{r})\delta(\sigma-\sigma_k)$, $Q_0(\mathbf{r}) = \sum_{k=1}^{S}\sigma_k\rho_k(\mathbf{r})$, where

$$\rho_k(\mathbf{r}) \equiv \frac{e^{\bar{\beta}\{\mu_k-\sigma_k[\Psi_0(\mathbf{r})-\alpha(\mathbf{r})]\}}}{\sum_{l=1}^{S} e^{\bar{\beta}\{\mu_l-\sigma_l[\Psi_0(\mathbf{r})-\alpha(\mathbf{r})]\}}}, \tag{17}$$

has spatial integral A_k, and is therefore the equilibrium number density for charge σ_k [14].

To illustrate properties of the solutions, consider the two level system, $\sigma_k = 0, \sigma_0$, in the beta-plane where $\alpha = \alpha_0 y$: $\rho_2 = (e^{\bar{\beta}[\sigma_0(\Psi_0-\alpha)-\mu]}+1)^{-1}$ is the Fermi function, where $\mu = \mu_1 - \mu_0$ is the chemical potential difference (this model, in a similar context, was also analyzed in detail in Ref. [11]). Define the temperature scale $\bar{T}_0 = -\sigma_0^2/4k_R^2 < 0$, and let $\bar{t} = \bar{T}/\bar{T}_0$, $p_0 = -(\sigma_0\tilde{\Psi}_0 - \mu)/2\bar{T}_0$, $q_0 = 2Q_0/\sigma_0 - 1$, $\boldsymbol{\rho} = (\rho_x, \rho_y) = k_R\mathbf{r}$. In terms of these scaled variables, the free energy and equilibrium equation take the form

$$\mathcal{G} = -\frac{\sigma_0^2}{4k_R^4}\int d^2\rho\left[\frac{1}{2}|\boldsymbol{\nabla}_\rho p_0|^2 - hp_0 + V(p_0) + \epsilon_0\right],$$

$$q_0 = (-\nabla_\rho^2 + 1)p_0 - h = \tanh(p_0/\bar{t}), \tag{18}$$

where $h(\boldsymbol{\rho}) = h_0 + g_0\rho_y$, with $h_0 = 1 - 2f_0/\sigma_0 + \mu/2\bar{T}_0$, $g_0 = -2(\alpha_0 k_R^2 + \beta_f)/\sigma_0 k_R$. The p_0-independent $\epsilon_0(\boldsymbol{\rho})$ term is unimportant. The potential $V(p_0) = \frac{1}{2}p_0^2 - \bar{t}\ln[2\cosh(p_0/\bar{t})]$ is an even function with a single minimum at $p_0 = 0$ for $\bar{t}>1$, and symmetric double minima at $\pm p_{\rm eq}^0(\bar{t})$ satisfying $p_{\rm eq}^0 = \tanh(p_{\rm eq}^0/\bar{t}) = q_{\rm eq}^0$ for $\bar{t}<1$.

Equation (18) is a standard continuum model of a binary fluid in a *gravitational field* g_0, composed of a mixture of heavy ($q_0 = 1$, $Q_0 = \sigma_0$) and light ($q_0 = -1$, $Q_0 = 0$) particles. The $|\boldsymbol{\nabla} p_0|^2$ term is an attractive interaction between like particles which encourages phase separation at low temperatures, $\bar{t}<1$. For $h\neq 0$, $V(p_0) - hp_0$ has a unique absolute minimum $p_{\rm eq}(\bar{t},h) = -p_{\rm eq}(\bar{t},-h)$. For $\bar{t}>1$, $p_e q$ is continuous through $h = 0$, while for $\bar{t}<1$ it jumps discontinuously between the heavy and light phases, $\pm p_{\rm eq}^0(\bar{t})$. This analogy makes it obvious that the stable equilibrium state must be vertically stratified, with p_0 and q_0 increasing monotonically (lighter phases floating on denser phases) in the direction of g_0 (southward if $\tilde{\beta}_f \equiv \beta_f + \alpha_0 k_R^2 > 0$). For $\bar{t}<1$ there will be a sharp interface centered at $\rho_y = -h_0/g_0$ (determined by μ) of

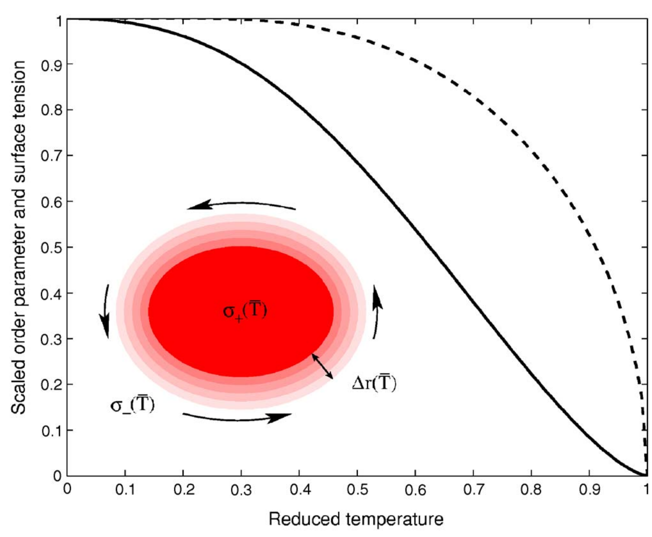

FIG. 1. (Color online) Scaled equilibrium charge density $q_{\rm eq}(\bar{t})$ (dashed line) and surface tension $\Sigma(\bar{t})$ (solid line) vs reduced temperature $\bar{t} = \bar{T}/\bar{T}_0$. For $\bar{t}\to 0$, $q_{\rm eq} \approx 1 - 2e^{-2/\bar{t}}$, $\Sigma \approx 1 - \pi^2\bar{t}^2/12$, while for $\bar{t}\to 1$, $q_{\rm eq} \approx \sqrt{3(1-\bar{t})}$, $\Sigma \approx [2(1-\bar{t})]^{3/2}$. Inset: schematic coherent vortex with asymptotic charge density $\sigma_\pm = \frac{1}{2}\sigma_0(1 \pm q_{\rm eq})$ inside and outside. Arrows indicate the direction of fluid flow, essentially parallel to the interface.

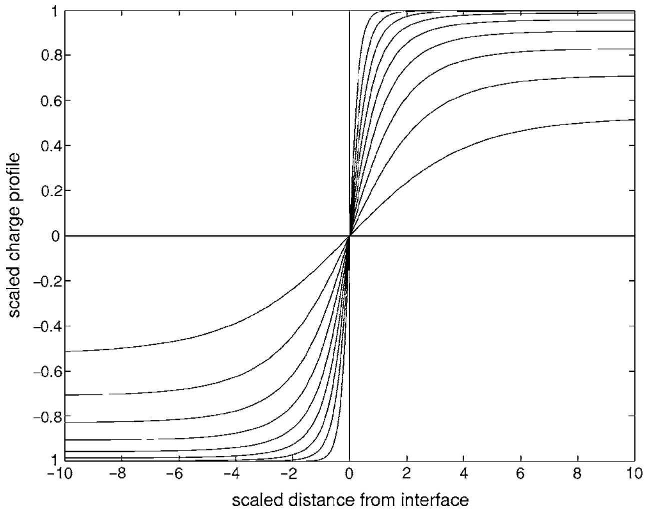

FIG. 2. Scaled interface profiles $q_0(\xi,\bar{t})$, as a function of the scaled normal coordinate ξ, for a sequence of reduced temperatures from $\bar{t}=0$ [discontinuous step, $q_0=\mathrm{sgn}(\xi)$] to $\bar{t}=1$ (completely flat, $q_0\equiv 0$) in steps of 0.1. For small $\bar{t}$, $q_0\approx\tanh(\xi/\bar{t})$ (hence width $\Delta r\approx\bar{t}/k_R$), while $p_0\approx\mathrm{sgn}(\xi)(1-e^{-|\xi|})$, remaining continuous even at $\bar{t}=0$. Near $\bar{t}=1$, $q_0\approx p_0\approx q_{\mathrm{eq}}\tanh(\xi/2\xi_0)$ with diverging width $\xi_0=k_R\Delta r=1/\sqrt{2(1-\bar{t})}$.

width $\Delta\rho_y\sim\bar{t}$ (see below) between the segregated phases. The analogy generalizes easily to multiple charge levels, which generate multicomponent fluid mixtures with different combinations of separated and unseparated phases produced as $\bar{t},\mu_i$ are varied. However, g_0 is unchanged, and in equilibrium the system must again be vertically stratified with phase density ordered along g_0. The argument also clearly generalizes to nonlinear, but monotonic $f(y)$: single signed $g_0(y)$ can only produce vertical stratification. This establishes the claim that only equilibria with purely zonal flow are stable in the presence of a beta effect, $\tilde{\beta}_f\neq 0$, and is completely consistent with inverse cascade arguments based on conservation of B.

Only if $\tilde{\beta}_f=0$, hence $\alpha_0=-\beta_f/k_R^2=\mathcal{O}(10\ \mathrm{cm/s})$ does the gravity effect disappear. In a background flow moving at speed α_0, an effective isotropic f plane is restored. Consider a large vortex region whose boundary is smooth on the scale of its width $\Delta r(\bar{T})$ (see below), viewed as a bubble of one phase entrained by another [15] (see inset to Fig. 1). Coexistence requires $\bar{T}<\bar{T}_0$, and $h=0$, i.e., $\mu=\sigma_0(\sigma_0-2f_0)/2k_R^2$. Near the interface, (18) is a 1D equation in the coordinate $\xi=\boldsymbol{\rho}\cdot\hat{\mathbf{n}}$ normal to the interface, with $q_0,p_0\to\pm q_{\mathrm{eq}}^0$ [hence, charges $\sigma_\pm(\bar{T})=\frac{1}{2}\sigma_0(1\pm q_{\mathrm{eq}}^0)$], deep on either side.

The free energy increment per unit length L of the interface, $\Delta\mathcal{G}/L=(2|\bar{T}_0|^{3/2}/\sigma_0)\Sigma$ yields the scaled surface tension $\Sigma(\bar{t})=\int_{-\infty}^{\infty}d\xi(\partial_\xi p_0)^2$. In Fig. 1 numerical solutions for q_{eq}^0, Σ are plotted (and exact asymptotics described), while in Fig. 2 scaled interface profiles $q_0(\xi)$ are plotted for several $\bar{t}$. The true equilibrium solution is finally obtained by minimizing the vortex perimeter L at fixed area A (set by the total charge Ω_1), yielding, not surprisingly, a circular vortex. These arguments again generalize to multiple charge levels, where one could in principle have more than two phases in simultaneous equilibrium, with vortices composed of a central core of one phase, ringed by one or more other phases.

Since B is no longer conserved in the f plane, there is no dynamical barrier to the formation of isotropic structures, again establishing consistency between arguments based on the dynamics of the inverse cascade, and those based on purely equilibrium thermodynamics. It appears likely that conservation of B is a microscopic reflection of the analogy to gravity-induced density stratification. Since vortices drift across g_0, rather than accelerating along it, equilibration is necessarily far less direct than in physical binary fluid systems, and B may be one of the mechanisms controlling this. This idea will be investigated more carefully in future work.

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[12] It follows from (1) that any dynamical variable $A[Q]$ obeys $\partial_t A=\{A,H\}$, in which $\{A,B\}=\int d^2r Q(\mathbf{r})\nabla\,\delta A/\delta Q(\mathbf{r})\times\nabla\,\delta B/\delta Q(\mathbf{r})$ is a Poisson bracket.
[13] Formally, one chooses λ in such a way that $\{Q,L\}=\partial_\xi Q$, where ξ is the symmetry coordinate, so L is the generator of translations along ξ.
[14] The *point charge* limit is obtained by letting all nonzero $\sigma_k\to\infty$, $A_k\to 0$ with fixed $q_k=\sigma_k A_k$. The temperature scales via $\bar{\beta}\sigma_k=\hat{\beta}q_k$ with finite $\hat{\beta}=1/\hat{T}$, and $\sigma_k\rho_k\to q_k\hat{\rho}_k$, with $\hat{\rho}_k=\exp\{\hat{\beta}[\hat{\mu}_k-q_k(\Psi_0-\alpha)]\}$, and $\hat{\mu}_k$ ensuring a unit integral. The nonzero charge region has measure zero. An example is the sinh-Poisson equation, $Q_0=C\sinh(\hat{\beta}\Psi_0)$, the symmetric special case of the three level system, $q_k=0,\pm 1$. F. Spineau and M. Vlad [Phys. Rev. Lett. **94**, 235003 (2005)] obtain, through a mysterious sequence of field theoretic mappings, an equilibrium equation with $Q_0=C_1\sinh(\hat{\beta}\Psi_0)[\cosh(\hat{\beta}\Psi_0)-C_2]$, the

symmetric special case of the five level system, $q_k = 0, \pm 1, \pm 2$. All of these are, in turn, very special cases of the general theory presented here.

[15] Since $\beta_f(y) \equiv f'(y)$ decreases with increasing $|y|$, $f(y) - \beta_f(y_0) \times (y - y_0) \approx f(y_0) - \frac{1}{2} f''(y_0)(y - y_0)^2$ is not perfectly flat, but has an extremum. In the northern (southern) hemisphere it traps lighter (denser) phases, while denser (lighter) phases will be pushed to higher and lower latitudes. Only lower (higher) charged coherent vortices may then be truly stable. Underlying topography may induce nonuniform zonal background flows that enhance such extrema, serving as models of Jupiter's Great Red Spot (Refs. [10,11]). The extremum also "squeezes" the entrained bubble, zonally elongating it, as observed on Jupiter.